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Newton's law in an effective non-commutative space–time

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Abstract
The Newtonian potential is computed exactly in a theory that is fundamentally non-commutative in the space–time coordinates. When the dispersion for the distribution of the source is minimal (i.e. it is equal to the non-commutative parameter θ), the behaviour for large and small distances is analysed.

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Introduction

Recently a new model of quantum field theory on non-commutative space–time, satisfying Lorentz invariance and unitarity, has been proposed in [1, 2]: it is shown that there is no need for the Weyl–Wigner–Moyal $\star$-product [3–5] and the non-commutativity is carried by a Gaussian cut-off in the Fourier transform of the fields. This is not an *ad hoc* regularization device but is a result coming from the averaging operation on coherent states. This cut-off (which depends on the non-commutative parameter θ [2]) is also present in the (Feynman) propagator and is responsible for the UV finiteness of the theory [1, 2, 6].

The aim of this paper is to compute the modification of the Newtonian potential due to the change of the Green's function caused by the non-commutativity of the space–time coordinates.

Newton's law

In order to deal with a theory that is effectively non-commutative in the space–time coordinates, we follow [1] and suppose that the propagator satisfies

$$-\Box_x G(x, x') = \left(\sqrt{\frac{a}{\pi}}\right)^4 \mathrm{e}^{-a(x-x')^2}, \tag{1}$$

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where $\Box_x = \partial_t^2 + \nabla_x^2$, $a = 1/4\theta$ and θ is the non-commutative parameter with the dimensions of an area[2]. Note that in the limit $\theta \to 0$ the Gaussian distribution becomes a delta function and standard commutative theory is recovered. As usual the Newtonian potential $V(x)$ is related to the fluctuation of the 00 component of the metric $h_{00}(x)$ through $V(x) = h_{00}(x)/2$ and the fluctuation is given by [7]

$$h_{00}(x) = -8\pi G_N \int \mathrm{d}^4x'\, G(x,x') T_{00}(x'), \tag{2}$$

where G_N is Newton's constant and T_{00} is the 00 component of the stress tensor of the source of the gravitational field. Since in a theory that is fundamentally non-commutative the concept of a point is meaningless, the source cannot be a delta function but will be given by a Gaussian distribution

$$T_{00}(x) = \int \frac{\mathrm{d}^3q}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{x}} f(q^2), \tag{3}$$

where $f(q^2) = \mathrm{e}^{-\alpha\vec{q}^2} M$, with α and M constants. We shall choose $\alpha = \theta$ because we want to consider a minimum dispersion for the source.

The Fourier transform of the propagator is called $G(k)$ and it is implicitly defined as

$$G(x,x') = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}k_\mu(x^\mu - x'^\mu)} G(k). \tag{4}$$

Considering that the Fourier transform of a Gaussian distribution is still a Gaussian function,

$$\left(\sqrt{\frac{1}{4\theta\pi}}\right)^4 \mathrm{e}^{-\frac{1}{4\theta}(x-x')^2} = \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}k_\mu(x-x')^\mu}\, \mathrm{e}^{-\theta(k_0^2+\vec{k}^2)}, \tag{5}$$

then $G(k)$ can be obtained through equation (1):

$$G(k) = \frac{\mathrm{e}^{-\theta(k_0^2+\vec{k}^2)}}{k_0^2 + \vec{k}^2}. \tag{6}$$

From equations (2)–(4) and (6) one can write

$$h_{00}(x) = -8\pi G_N \int \mathrm{d}^4x' \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \mathrm{e}^{\mathrm{i}k_\mu(x^\mu - x'^\mu)} \frac{\mathrm{e}^{-\theta(k_0^2+\vec{k}^2)}}{k_0^2+\vec{k}^2} \int \frac{\mathrm{d}^3q}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\vec{q}\cdot\vec{x}'} f(q^2). \tag{7}$$

Since the source does not depend on time we can use

$$\int \mathrm{d}t'\, \mathrm{e}^{-\mathrm{i}k_0 t'} = 2\pi\,\delta(k_0), \tag{8}$$

and with the help of

$$\int \mathrm{d}^3x'\, \mathrm{e}^{\mathrm{i}\vec{x}'\cdot(\vec{k}-\vec{q})} = (2\pi)^3 \delta(\vec{k} - \vec{q}), \tag{9}$$

it is possible to simplify (7), obtaining the following equation:

$$h_{00}(x) = -8\pi G_N \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{x}} \frac{\mathrm{e}^{-\theta\vec{k}^2}}{\vec{k}^2} f(k^2). \tag{10}$$

Performing the integration over the angular part of d^3k one finds

$$h_{00}(r) = -\frac{2G_N}{\pi}\frac{M}{\mathrm{i}r}\int_{-\infty}^{\infty} \mathrm{d}k\, k\, \mathrm{e}^{\mathrm{i}kr} \frac{\mathrm{e}^{-(\theta+\alpha)k^2}}{k^2+\epsilon^2}, \tag{11}$$

[2] We are considering an Euclidean signature of the metric, coming from a Wick rotation of the time coordinate.

where the definition of $f(k^2)$ has been used, with $k = \sqrt{k^2}$, and the limit $\epsilon \to 0$ is understood[3]. The integration can be performed exactly [9] and the Newtonian potential is found to be the following:

$$V(r) = -G_N \frac{M}{r} \operatorname{Erf}\left(\frac{r}{2\sqrt{\theta + \alpha}}\right), \tag{12}$$

where $V(r) = h_{00}(r)/2$ has been used.

First of all we note that if $\alpha \gg \theta$ then the non-commutativity is screened by the source. This is natural since the source must be localized as much as possible in space. But in a theory that is fundamentally non-commutative, the smallest spread is given by θ.

We set now $\alpha = \theta$ since we wish to analyse the minimum case. In the region for which $r \gg 2\sqrt{2\theta}$ one obtains

$$V(r) = -G_N \frac{M}{r}\left[1 + \mathrm{e}^{-r^2/(8\theta)}\left(-\frac{2}{\sqrt{\pi}}\frac{\sqrt{2\theta}}{r} + \mathcal{O}\left(\frac{2\sqrt{2\theta}}{r}\right)^3\right)\right]. \tag{13}$$

Numerically (on requiring that the correction is of order 1) one finds that the large distance correction becomes important for the following critical value:

$$\frac{r_c}{2\sqrt{2\theta}} \simeq 0.4576. \tag{14}$$

Since it is known that Newton's law is verified up to a distance of the order of 200 μm [8], it is then possible to find a constraint for θ, $\theta < 10^{-8}m^2$. Of course due to the fact that Newton's law is not tested for very small distances, this constraint is not so strong. One has to use precision measurements in order to have a significant bound. In the opposite region where $r \ll 2\sqrt{2\theta}$ one obtains

$$V(r) = -G_N \frac{M}{r}\left[\frac{r}{\sqrt{\pi(2\theta)}} + \mathcal{O}\left(\frac{r}{2\sqrt{2\theta}}\right)^3\right]. \tag{15}$$

It is interesting to note that there is no divergence (unlike the commutative case): the introduction of the θ parameter gives a minimal length that regularizes the theory

$$V(0) = -G_N \frac{M}{\sqrt{\pi(2\theta)}}. \tag{16}$$

Conclusions

The main result of this brief paper is given by equation (12). The calculation is exact and no approximation has been made. We note that there is a deviation from the standard Newtonian potential but the shape of the source can screen the effect of non-commutative space. In the minimal case (i.e. the dispersion for the profile of the source is given by θ) such a deviation has been computed for the large and small distance approximations. In the first case we used the corrections to Newton's law in order to constrain the non-commutative parameter (but the obtained bound is weak since Newton's law is verified down to distances of the order of 200 μm). In the second case it is seen that there is a regular behaviour at $r = 0$. This reflects the well-known fact that non-commutativity introduces a minimal length which regularizes the theory.

[3] As usual this limit has been introduced in order to regularize the integrand at $k^2 = 0$.

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References

[1] Smailagic A and Spallucci E 2003 UV divergence-free QFT on noncommutative plane *J. Phys. A: Math. Gen.* **36** L517–21 (*Preprint* hep-th/0308193)

[2] Smailagic A and Spallucci E 2004 Lorentz invariance, unitarity in UV finite of QFT on noncommutative spacetime *J. Phys. A: Math. Gen.* **37** 1–10 (*Preprint* hep-th/0406174)
Smailagic A and Spallucci E 2004 Lorentz invariance, unitarity in UV finite of QFT on noncommutative spacetime *J. Phys. A: Math. Gen.* **37** 7169 (erratum) (*Preprint* hep-th/0406174)

[3] Weyl H 1927 Quantum mechanics and group theory *Z. Phys.* **46** 1

[4] Wigner E P 1932 On the quantum correction for thermodynamic equilibrium *Phys. Rev.* **40** 749–60

[5] Moyal J E 1949 Quantum mechanics as a statistical theory *Proc. Camb. Phil. Soc.* **45** 99–124

[6] Nicolini P 2004 Vacuum energy momentum tensor in (2+1) NC scalar field theory *Preprint* hep-th/0401204

[7] Wald R M 1984 *General Relativity* (Chicago, IL: Chicago University Press)

[8] Hoyle C D, Schmidt U, Heckel B R, Adelberger E G, Gundlach J H, Kapner D J and Swanson H E 2001 Submillimeter tests of gravitational inverse square law: a search for ‘large’ extra dimensions *Phys. Rev. Lett.* **86** 1418–21 (*Preprint* hep-ph/0011014)

[9] Gradshteyn I S and Ryzhik I M 1965 *Table of Integrals, Series and Products* (New York: Academic)